MFC CDT Probability and Statistics
Week 9

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Mathematics for our Future Climate: Theory, Data and Simulation (MFC CDT).

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IMPERIAL

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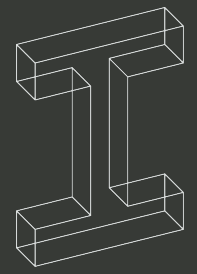
Recall our basic task:



Recall our basic task:

- We wanted to sample from distributions $\pi(x) \propto \gamma(x)$ given only the knowledge of $\gamma(x)$ and use these samples to estimate an integral

$$(\varphi, \pi) = \int \varphi(x)\pi(x)\,\mathrm{d}x$$

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- We will have a *sequence* of distributions $(\pi)_{t\geq 0}$ and we will want to estimate the integrals

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This will include the filtering problem.

State-space models

problem definition

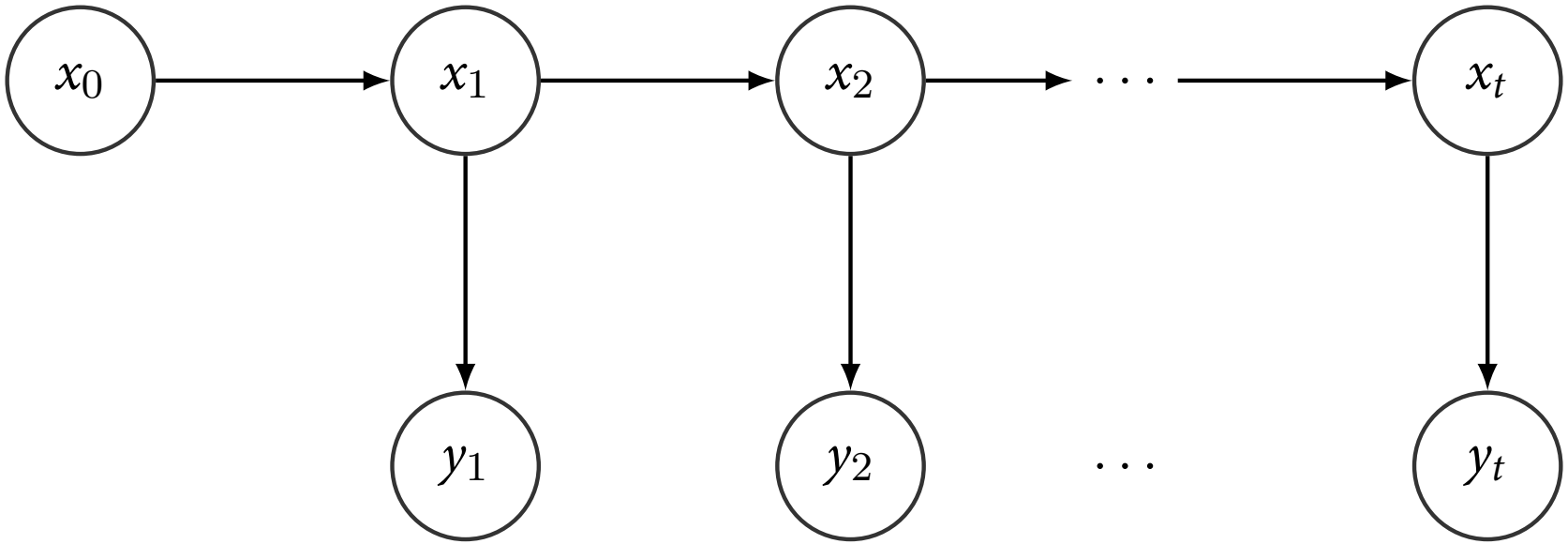

Figure: The conditional independence structure of a state-space model.

$(x_t)_{t\in\mathbb{N}_+}$: *hidden* signal process, $(y_t)_{t\in\mathbb{N}_+}$ the observation process.

$$x_0 \sim \pi_0(\mathrm{d}x_0), \qquad \text{(prior distribution)}$$
$$x_t|x_{t-1} \sim \tau_t(\mathrm{d}x_t|x_{t-1}), \qquad \text{(transition model)}$$
$$y_t|x_t \sim g_t(y_t|x_t), \qquad \text{(likelihood)}$$

$x_t \in \mathrm{X}$ where X is the state-space. We use: $g_t(x_t) = g_t(y_t|x_t)$.

State-space models

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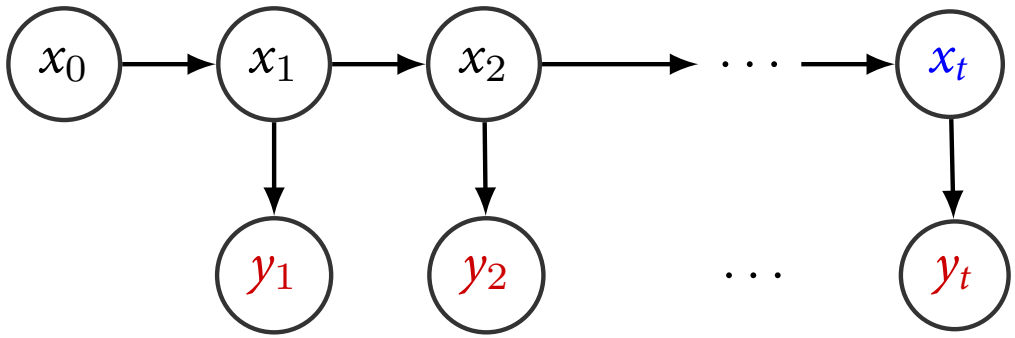
We are interested estimating expectations,

$$(\varphi, \pi_t) = \int \varphi(x_t)\pi(x_t|y_{1:t})\mathrm{d}x_t = \int \varphi(x_t)\pi_t(\mathrm{d}x_t),$$

sequentially as new data arrives. This problem is known as *the filtering problem*.



A simpler problem

Sequential inference

Let us first consider a generic probabilistic setting,

$$\pi_0(x) \qquad \text{and} \qquad g_t(y_t|x).$$

for $(y_t)_{t\in\mathbb{N}_+}$ a sequence of observations.

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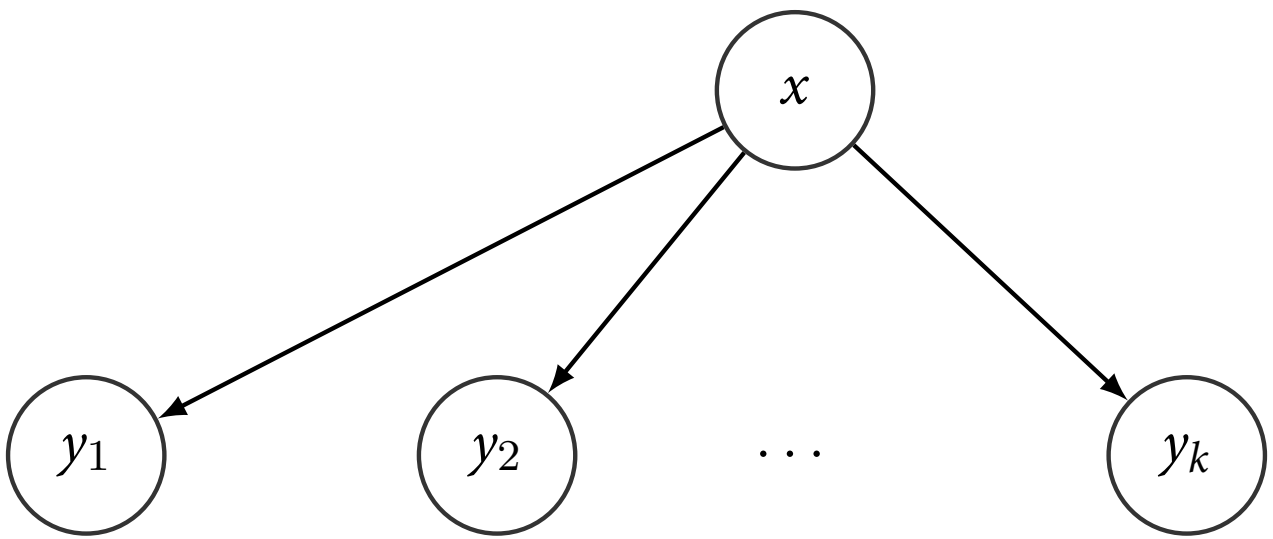

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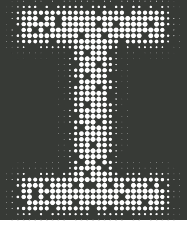

Mini-quiz: How would you obtain $\pi(x|y_{1:t})$?

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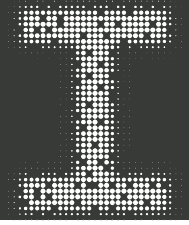

Mini-quiz: How would you obtain $\pi(x|y_{1:t})$?

We can use Bayes' rule iteratively

$$\begin{aligned}\pi(x|y_{1:t}) &= \frac{\gamma(x, y_{1:t})}{p(y_{1:t})}, \\ &= \frac{g_t(y_t|x)\gamma(x, y_{1:t-1})}{p(y_t|y_{1:t-1})p(y_{1:t-1})}, \\ &= \frac{g_t(y_t|x)\pi(x|y_{1:t-1})}{p(y_t|y_{1:t-1})}.\end{aligned}$$

where

$$p(y_t|y_{1:t-1}) = \int g_t(y_t|x)\pi(x|y_{1:t-1})\mathrm{d}x.$$

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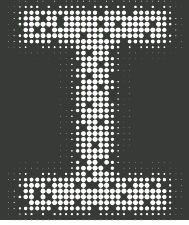

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where

$$p(y_t|y_{1:t-1}) = \int g_t(y_t|x)\pi(x|y_{1:t-1})\mathrm{d}x.$$

The previous posterior $\pi(x|y_{1:t-1})$ is used as the prior for the next step.

A simpler problem

Sequential inference: the Gaussian case

Let us assume that

$$\pi_0(x) = \mathcal{N}(x; \mu_0, V_0),$$
$$g_t(y_t|x) = \mathcal{N}(y_t; H_t x, R_t).$$

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Lemma 1

We obtain $\pi(x|y_{1:t}) = \mathcal{N}(x; \mu_t, V_t)$ *where,*

$$\mu_t = \mu_{t-1} + V_{t-1}H_t^\top (R_t + H_t V_{t-1} H_t^\top)^{-1}(y_t - H_t \mu_{t-1}),$$
$$V_t = V_{t-1} - V_{t-1}H_t^\top (R_t + H_t V_{t-1} H_t^\top)^{-1} H_t V_{t-1},$$

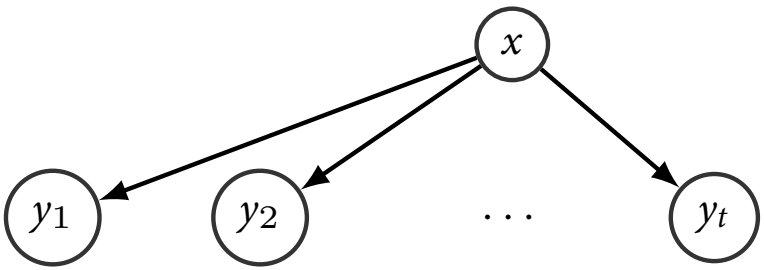
for $t \geq 1$.

Static vs. dynamic setting

Static inference: Given a probability model,

$$x \sim \pi_0(\mathrm{d}x),$$
$$y_t|x_t \sim g_t(y_t|x),$$

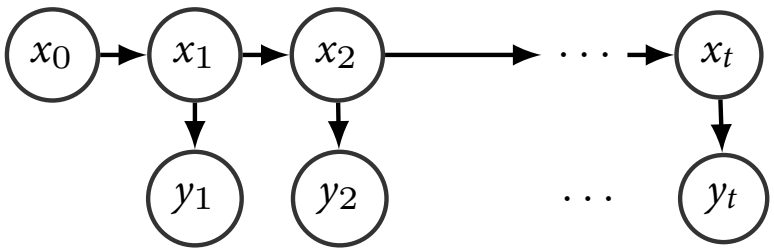
we are interested in static inference: Estimating $\pi(x|y_{1:t})$ sequentially.



Dynamic inference: Given a SSM,

$$x_0 \sim \pi_0(\mathrm{d}x_0),$$
$$x_t|x_{t-1} \sim \tau_t(\mathrm{d}x_t|x_{t-1}),$$
$$y_t|x_t \sim g_t(y_t|x_t),$$

we are interested in *the stochastic filtering problem*: Estimating $\pi_t(x_t|y_{1:t})$.



State-space models

Algorithmic principle

We are interested in estimating expectations,

$$(\varphi, \pi_t) = \int \varphi(x_t)\pi_t(x_t|y_{1:t})\mathrm{d}x_t = \int \varphi(x_t)\pi_t(\mathrm{d}x_t),$$

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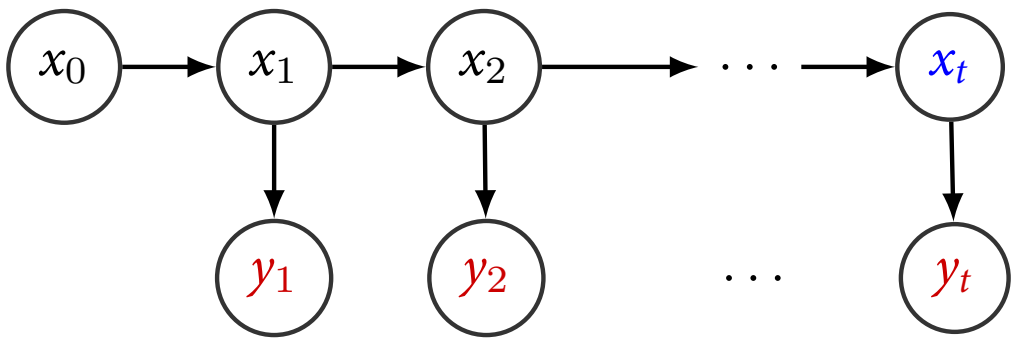

Algorithm:

Predict

$$\xi_t(\mathrm{d}x_t) = \int \pi_{t-1}(\mathrm{d}x_{t-1})\tau_t(\mathrm{d}x_t|x_{t-1})$$

Update

$$\pi_t(\mathrm{d}x_t) = \xi_t(\mathrm{d}x_t)\frac{g_t(y_t|x_t)}{p(y_t|y_{1:t-1})}.$$

Let us look in detail to these steps:

Prediction: Given $\pi_{t-1}(\mathrm{d}x_{t-1}|y_{1:t-1})$, we want to compute $\pi_t(\mathrm{d}x_t|y_{1:t-1})$.

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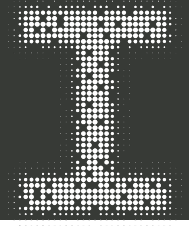
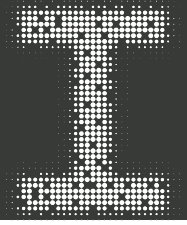
In terms of densities

$$\pi_t(x_t|y_{1:t-1}) = \int \pi_{t-1}(x_{t-1}|y_{1:t-1})\tau_t(x_t|x_{t-1})\mathrm{d}x_{t-1}.$$



We have already seen the update rule, but we modify this in the dynamic setting: Our prior will now be the predictive distribution $\pi_t(\mathrm{d}x_t|y_{1:t-1})$.



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$$\begin{aligned}\pi_t(x_t|y_{1:t}) &= \frac{\gamma(x_t, y_{1:t})}{p(y_{1:t})}, \\ &= \frac{g_t(y_t|x_t)\pi_t(x_t|y_{1:t-1})}{p(y_t|y_{1:t-1})}.\end{aligned}$$

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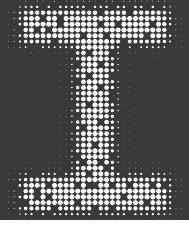
$$\begin{aligned}\pi_0(x) &= \mathcal{N}(x; \mu_0, V_0),\\ \tau_t(x_t|x_{t-1}) &= \mathcal{N}(x_t; A_t x_{t-1}, Q_t),\\ g_t(y_t|x_t) &= \mathcal{N}(y_t; H_t x_t, R_t).\end{aligned}$$



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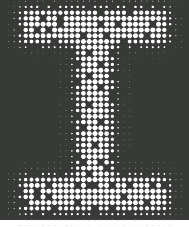
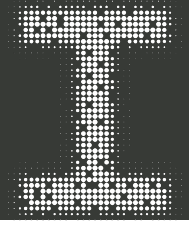


Yes, we can!



Yes, we can!

Lemma 2

Given the optimal filter $\pi_{t-1}(x_{t-1}|y_{1:t-1}) = \mathcal{N}(x_{t-1}; \mu_{t-1}, V_{t-1})$ *at time* $t-1$ *the predictive distribution* $\xi_t(x_t|y_{1:t-1})$ *is given by*

$$\xi_t(x_t|y_{1:t-1}) = \mathcal{N}(x_t; \tilde{\mu}_t, \tilde{V}_t),$$

where,

$$\tilde{\mu}_t = A_t \mu_{t-1}, \tag{1}$$

$$\tilde{V}_t = A_t V_{t-1} A_t^\top + Q_t. \tag{2}$$

Lemma 3

Finally, the optimal filter $\pi_t(x_t|y_{1:t})$ *is given by*

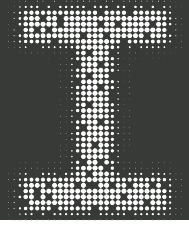
$$\pi_t(x_t|y_{1:t-1}) = \mathcal{N}(x_t; \mu_t, V_t),$$

where,

$$\mu_t = \tilde{\mu}_t + \tilde{V}_t H_t^\top (R_t + H_t \tilde{V}_t H_t^\top)^{-1}(y_t - H_t \tilde{\mu}_t), \tag{3}$$

$$V_t = \tilde{V}_t - \tilde{V}_t H_t^\top (R_t + H_t \tilde{V}_t H_t^\top)^{-1} H_t \tilde{V}_t, \tag{4}$$

from Lemma 1.



What if nonlinearities exist in Gaussian models?

$$\begin{aligned}\pi_0(x) &= \mathcal{N}(x; \mu_0, V_0),\\ \tau_t(x_t|x_{t-1}) &= \mathcal{N}(x_t; a_t(x_{t-1}), Q_t),\\ g_t(y_t|x_t) &= \mathcal{N}(y_t; h_t(x_t), R_t).\end{aligned}$$

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Can we still do analytical computations?

Yes! We can use the *extended Kalman filter* (EKF) or the *unscented Kalman filter* (UKF).

Assume that we are given the SSM

$$\begin{aligned}\pi_0(x_0) &= \mathcal{N}(x_0; \mu_0, V_0),\\ \tau_t(x_t|x_{t-1}) &= \mathcal{N}(x_t; a_t(x_{t-1}), Q_t)\\ g_t(y_t|x_t) &= \mathcal{N}(y_t; h_t(x_t), R_t).\end{aligned}$$

where $a_t : \mathrm{X} \to \mathrm{X}$, $h_t : \mathrm{X} \to \mathrm{Y}$, $Q_t \in \mathbb{R}^{d_x \times d_x}$, and $R_t \in \mathbb{R}^{d_y \times d_y}$. Assume that the approximate posterior distribution at time $t-1$ is $\pi^E_{t-1}(x_{t-1}) = \mathcal{N}(x_{t-1}; \mu^E_{t-1}, V^E_{t-1})$.

If the model is approximately locally linear, one can linearize $a_t(x_t)$ around μ^E_{t-1} and obtain the dynamical model

$$\bar{a}_t(x_t) = a_t(\mu^E_{t-1}) + A_t(x_t - \mu^E_{t-1}) = a_t(\mu^E_{t-1}) + A_t x_t - A_t \mu^E_{t-1}, \tag{5}$$

where

$$A_t = \left.\frac{\partial a_t(x)}{\partial x}\right|_{x=\mu^E_{t-1}}.$$

We can see (5) as a linear model with control inputs. Hence, the prediction step with this linearized model simply becomes

$$\tilde{\mu}^E_t = a_t(\mu^E_{t-1}).$$

The uncertainty is propagated also as in the KF, since (5) is a linear model, hence we obtain

$$\tilde{V}_t^E = A_t V_{t-1}^E A_t^\top + Q_t.$$

Similarly, given $\tilde{\mu}_t^E$, in order to proceed with the observation model we can linearize h_t around $\tilde{\mu}_t^E$, i.e., we construct

$$\bar{h}_t(x_t) = h_t(\tilde{\mu}_t^E) + H_t(x_t - \tilde{\mu}_t^E),$$

where

$$H_t = \left.\frac{\partial h_t(x)}{\partial x}\right|_{x=\tilde{\mu}_t}.$$

Given the linearization, the EKF update step now becomes

$$\mu_t^E = \tilde{\mu}_{t-1}^E + \tilde{V}_t^E H_t^\top (R_t + H_t \tilde{V}_t^E H_t^\top)^{-1}(y_t - h_t(\tilde{\mu}_t^E)),$$
$$V_t^E = \tilde{V}_t^E - \tilde{V}_t^E H_t^\top (R_t + H_t \tilde{V}_t^E H_t^\top)^{-1} H_t \tilde{V}_t^E.$$

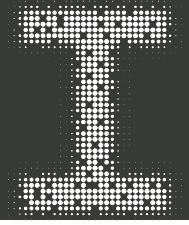
Finally, one can compactly summarize the EKF as follows. Given $\pi^E_{t-1}(x_{t-1}) = \mathcal{N}(x_{t-1}; \mu^E_{t-1}, V^E_{t-1})$, the new posterior pdf $\pi^E_t(x_t) = \mathcal{N}(x_t; \mu^E_t, V^E_t)$ is obtained via

$$\tilde{\mu}^E_t = a_t(\mu^E_{t-1}), \tag{6}$$

$$\tilde{V}^E_t = A_t V^E_{t-1} A_t^\top + Q_t, \tag{7}$$

$$\mu^E_t = \tilde{\mu}^E_{t-1} + \tilde{V}^E_t H_t^\top (R_t + H_t \tilde{V}^E_t H_t^\top)^{-1}(y_t - h_t(\tilde{\mu}^E_t)), \tag{8}$$

$$V^E_t = \tilde{V}^E_t - \tilde{V}^E_t H_t^\top (R_t + H_t \tilde{V}^E_t H_t^\top)^{-1} H_t \tilde{V}^E_t. \tag{9}$$



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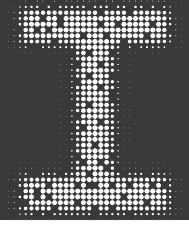
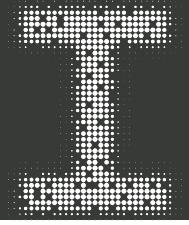
Many other variants, very popular in fields like robotics, navigation, guidance, aerospace, finance, vision, etc.

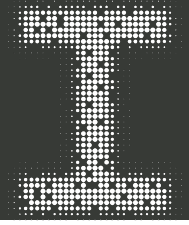


We can use the *unscented transform* to obtain a Gaussian approximation of the posterior distribution.



We can use the *unscented transform* to obtain a Gaussian approximation of the posterior distribution.

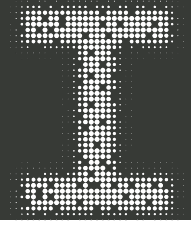
Let X be a Gaussian random variable $X \sim \mathcal{N}(x; \mu, \Sigma)$.



We can use the *unscented transform* to obtain a Gaussian approximation of the posterior distribution.

Let X be a Gaussian random variable $X \sim \mathcal{N}(x; \mu, \Sigma)$.

Say we would like to compute moments of $g(X)$ where g is a nonlinear function.



The unscented transform precisely approximates the moments of $g(X)$ by the moments of a Gaussian random variable.

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The unscented transform is based on the idea of *sigma points*, which are chosen deterministically:

$$\begin{aligned}
\sigma_0 &= \mu, \\
\sigma_i &= \mu + \left(\sqrt{(n+\lambda)\Sigma}\right)_i, \quad i = 1, \ldots, n, \\
\sigma_i &= \mu - \left(\sqrt{(n+\lambda)\Sigma}\right)_{i-n}, \quad i = n+1, \ldots, 2n,
\end{aligned}$$

where $\lambda = \alpha^2(n+\kappa) - n$ and α and κ are parameters that can be chosen freely.

How is this idea used given:

$$\begin{aligned}
\pi_0(x_0) &= \mathcal{N}(x_0; \mu_0, V_0), \\
\tau_t(x_t|x_{t-1}) &= \mathcal{N}(x_t; a_t(x_{t-1}), Q_t) \\
g_t(y_t|x_t) &= \mathcal{N}(y_t; h_t(x_t), R_t).
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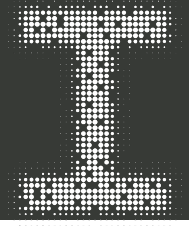
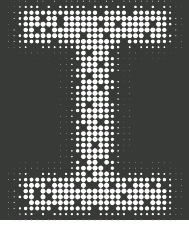
Given μ_t, V_t, use the nonlinearity $h_t(\cdot)$ using unscented transform to compute the moments of updated posterior.



We have been looking at the filtering problem, i.e., estimating $\pi_t(x_t|y_{1:t})$.



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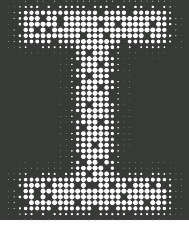
What if we want to estimate $\pi_t(x_t|y_{1:T})$ for $T > t$?



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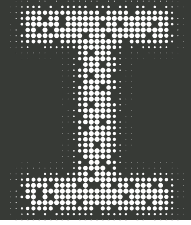
What if we want to estimate $\pi_t(x_t|y_{1:T})$ for $T > t$?

This is called the *smoothing problem*. These methods are usually implemented backwards in time.



We have smoothing recursions

$$\pi(x_{t+1}|y_{1:t}) = \int \tau(x_{t+1}|x_t)\pi(x_t|y_{1:t})\mathrm{d}x_t,$$

$$\pi(x_t|y_{1:T}) = \pi(x_t|y_{1:t}) \int \frac{\tau(x_{t+1}|x_t)\pi(x_{t+1}|y_{1:T})}{\pi(x_{t+1}|y_{1:t})}\mathrm{d}x_{t+1}.$$

Proof: Let us notice

$$\begin{aligned} p(x_t|x_{t+1}, y_{1:T}) &= p(x_t|x_{t+1}, y_{1:t}), \\ &= \frac{p(x_t, x_{t+1}|y_{1:t})}{p(x_{t+1}|y_{1:t})}, \\ &= \frac{\pi(x_t|y_{1:t})\tau(x_{t+1}|x_t)}{\pi(x_{t+1}|y_{1:t})}, \end{aligned}$$

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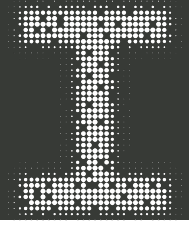
where the last equality follows from the Markov property. Now we construct the joint

$$\begin{aligned} p(x_{t+1}, x_t|y_{1:T}) &= p(x_t|x_{t+1}, y_{1:T})p(x_{t+1}|y_{1:T}), \\ &= \frac{\pi(x_t|y_{1:t})\tau(x_{t+1}|x_t)}{\pi(x_{t+1}|y_{1:t})}\pi(x_{t+1}|y_{1:T}). \end{aligned}$$

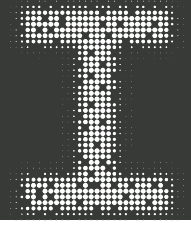
By integrating out x_{t+1}, the result follows.



Let us consider our linear-Gaussian model again

$$\begin{aligned}\pi_0(x) &= \mathcal{N}(x; \mu_0, V_0),\\ \tau_t(x_t|x_{t-1}) &= \mathcal{N}(x_t; A_t x_{t-1}, Q_t),\\ g_t(y_t|x_t) &= \mathcal{N}(y_t; H_t x_t, R_t).\end{aligned}$$

In this setting, smoothing can be exactly implemented too.



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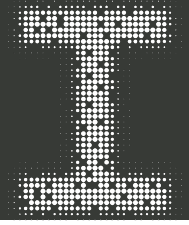
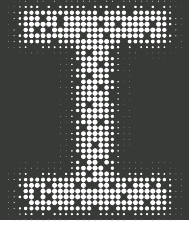
The resulting algorithm is called the *Rauch-Tung-Striebel* (RTS) smoother.



Assume we have computed filter moments $(\mu_t, V_t)_{t=0}^T$.



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$$\begin{aligned}
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\mu_t^s &= \mu_t + J_t(\mu_{t+1}^s - A_t\mu_t), \\
V_t^s &= V_t + J_t(V_{t+1}^s - V_t)J_t^\top,
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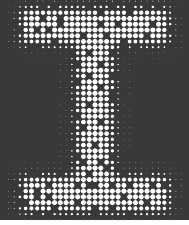
where

$$J_t = V_t A_t^\top \hat{V}_{t+1}^{-1}.$$



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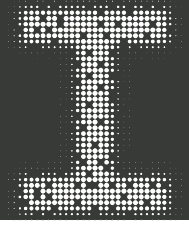

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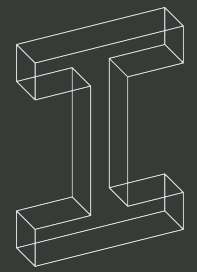

Deterministic approximations are only useful in certain settings where we can ensure

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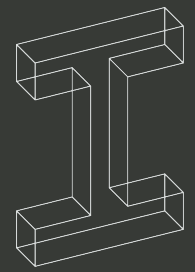
- Gaussianity



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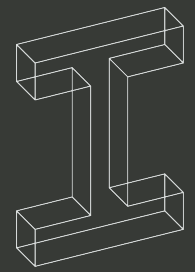
We will now introduce a general Monte Carlo approach to estimate posterior distributions $\pi_t^N(\mathrm{d}x_t|y_{1:t})$.



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Particle filters.

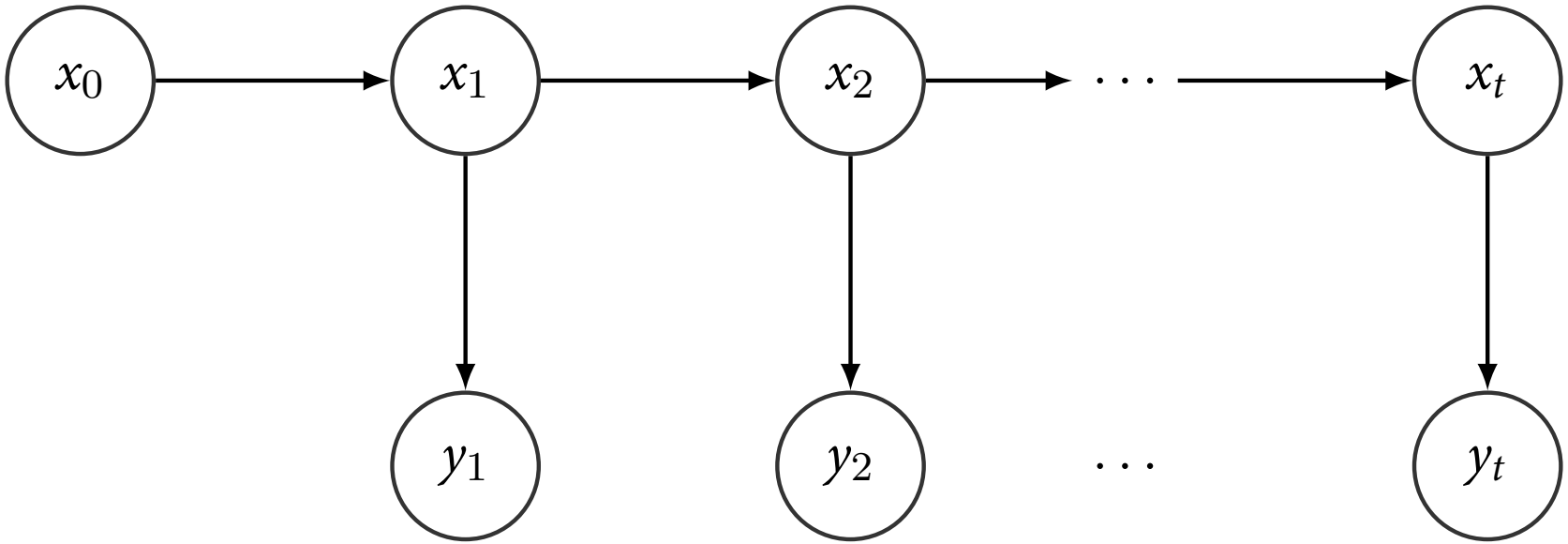
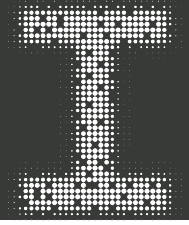


Figure: The conditional independence structure of a state-space model.

$(x_t)_{t\in\mathbb{N}_+}$: *hidden* signal process, $(y_t)_{t\in\mathbb{N}_+}$ the observation process.

$$x_0 \sim \pi_0(\mathrm{d}x_0), \qquad \text{(prior distribution)}$$
$$x_t|x_{t-1} \sim \tau_t(\mathrm{d}x_t|x_{t-1}), \quad \text{(transition model)}$$
$$y_t|x_t \sim g_t(y_t|x_t), \qquad \text{(likelihood)}$$

$x_t \in \mathrm{X}$ where X is the state-space. We use: $g_t(x_t) = g_t(y_t|x_t)$.



Before we go into the details of the derivation, let us directly look at the algorithm.

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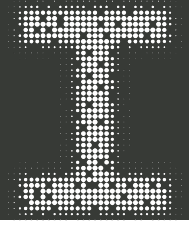
- Sampling: draw
$$\bar{x}_t^{(i)} \sim \tau_t(\mathrm{d}x_t | x_{t-1}^{(i)})$$
independently for every $i = 1, \ldots, N$.
- Weighting: compute
$$w_t^{(i)} = g_t(\bar{x}_t^{(i)}) / \bar{Z}_t^N$$
for every $i = 1, \ldots, N$, where $\bar{Z}_t^N = \sum_{i=1}^N g_t(\bar{x}_t^{(i)})$.
- Resampling: draw independently,
$$x_t^{(i)} \sim \tilde{\pi}_t(\mathrm{d}x) := \sum_i w_t^{(i)} \delta_{\bar{x}_t^{(i)}}(\mathrm{d}x) \quad \text{for } i = 1, \ldots, N.$$

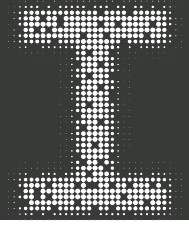
$$\pi_{t-1}^N \underbrace{\rightarrow}_{\text{sampling}} \xi_t^N \underbrace{\rightarrow}_{\text{weighting}} \tilde{\pi}_t^N \underbrace{\rightarrow}_{\text{resampling}} \pi_t^N.$$

Particle filters

Derivation



Where does the algorithm come from?



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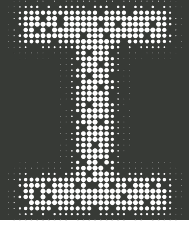
Surprisingly, we will not use the prediction-update recursions directly unlike in the Kalman filter.



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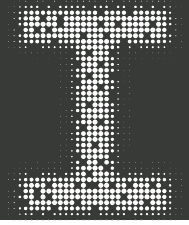
Surprisingly, we will not use the prediction-update recursions directly unlike in the Kalman filter.

We will instead develop an importance sampler on the path space.



The key recursion on the path distributions:

$$\begin{aligned}
\pi_t(x_{0:t}|y_{1:t}) &= \frac{\gamma(x_{0:t}, y_{1:t})}{p(y_{1:t})} \\
&= \frac{\gamma(x_{0:t-1}, y_{1:t-1})}{p(y_{1:t-1})} \frac{\tau(x_t|x_{t-1})g(y_t|x_t)}{p(y_t|y_{1:t-1})} \\
&= \pi_t(x_{0:t-1}|y_{1:t-1}) \frac{\tau(x_t|x_{t-1})g(y_t|x_t)}{p(y_t|y_{1:t-1})}.
\end{aligned}$$

Recall importance sampling: Assume that we aim at estimating expectations of a given density π, i.e., we would like to compute

$$(\varphi, \pi) = \int \varphi(x)\pi(x)\mathrm{d}x.$$

We also assume that sampling from this density is not possible and we can only evaluate the *unnormalised* density $\gamma(x)$.

One way to estimate this expectation is to sample from a proposal measure q and rewrite the integral as

$$\begin{aligned}(\varphi, \pi) &= \int \varphi(x)\pi(x)\mathrm{d}x, \\ &= \frac{\int \varphi(x)\frac{\gamma(x)}{q(x)}q(x)\mathrm{d}x}{\int \frac{\gamma(x)}{q(x)}q(x)\mathrm{d}x}, \\ &\approx \frac{\frac{1}{N}\sum_{i=1}^{N}\varphi(x^{(i)})\frac{\gamma(x^{(i)})}{q(x^{(i)})}}{\frac{1}{N}\sum_{i=1}^{N}\frac{\gamma(x^{(i)})}{q(x^{(i)})}}, \qquad x^{(i)} \sim q, \quad i = 1, \ldots, N. \quad (10)\end{aligned}$$

Let us now introduce the unnormalised weight function

$$W(x) = \frac{\gamma(x)}{q(x)}. \tag{11}$$

With this, the Eq. (10) becomes

$$\begin{aligned}(\varphi, \pi^N) &= \frac{\frac{1}{N}\sum_{i=1}^N \varphi(x^{(i)})W(x^{(i)})}{\frac{1}{N}\sum_{i=1}^N W(x^{(i)})}, \qquad x^{(i)} \sim q, \quad i = 1, \ldots, N, \\ &= \frac{\sum_{i=1}^N \varphi(x^{(i)})\mathrm{W}^{(i)}}{\sum_{i=1}^N \mathrm{W}^{(i)}}, \qquad x^{(i)} \sim q, \quad i = 1, \ldots, N,\end{aligned}$$

where $\mathrm{W}^{(i)} = W(x^{(i)})$ are called *the unnormalised weights*.

Finally, we can obtain the estimator in a more convenient form,

$$(\varphi, \pi^N) = \sum_{i=1}^{N} \mathrm{w}^{(i)} \varphi(x^{(i)}).$$

by introducing the *normalised importance weights*

$$\mathrm{w}^{(i)} = \frac{\mathrm{w}^{(i)}}{\sum_{i=1}^{N} \mathrm{w}^{(i)}}, \tag{12}$$

for $i = 1, \ldots, N$. We note that the particle approximation of π in this case is given as

$$\pi^N(\mathrm{d}x) = \sum_{i=1}^{N} \mathrm{w}^{(i)} \delta_{x^{(i)}}(\mathrm{d}x). \tag{13}$$

In the following, we will derive the importance sampler aiming at building particle approximations of $\pi_t(x_{0:t}|y_{1:t})$ for a state-space model.

The proposal over the entire path space $x_{0:t}$ denoted $q(x_{0:t})$. Note

$$\gamma(x_{0:t}, y_{1:t}) = \mu(x_0) \prod_{k=1}^{t} \tau(x_k|x_{k-1}) g(y_k|x_k). \tag{14}$$

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This simply the joint distribution of all variables $(x_{0:t}, y_{1:t})$. Just as in the regular importance sampling

$$W_{0:t}(x_{0:t}) = \frac{\gamma(x_{0:t}, y_{1:t})}{q(x_{0:t})}.$$

Obviously, given samples from the proposal $x_{0:t}^{(i)} \sim q(x_{0:t})$, by evaluating the weight $\mathrm{W}_{0:t}^{(i)} = W_{0:t}(x_{0:t}^{(i)})$ for $i = 1, \ldots, N$ and building a particle approximation

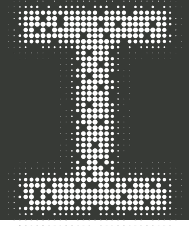
$$\pi^N(\mathrm{d}x_{0:t}) = \sum_{i=1}^{N} \mathrm{W}_{0:t}^{(i)} \delta_{x_{0:t}^{(i)}}(\mathrm{d}x_{0:t}).$$

Let us consider a decomposition of the proposal

$$q(x_{0:t}) = q(x_0) \prod_{k=1}^{t} q(x_k|x_{1:k-1}).$$

Note that, based on this, we can build a recursion for the function $W(x_{0:t})$ by writing

$$\begin{aligned} W_{0:t}(x_{0:t}) &= \frac{\gamma(x_{0:t}, y_{1:t})}{q(x_{0:t})}, \\ &= \frac{\gamma(x_{0:t-1}, y_{1:t-1})}{q(x_{0:t-1})} \frac{\tau(x_t|x_{t-1})g(y_t|x_t)}{q(x_t|x_{0:t-1})}, \\ &= W_{0:t-1}(x_{0:t-1}) \frac{\tau(x_t|x_{t-1})g(y_t|x_t)}{q(x_t|x_{0:t-1})}, \\ &= W_{0:t-1}(x_{0:t-1}) W_t(x_{0:t}). \end{aligned} \tag{15}$$



This is still not optimal, as we still need to store the whole path.

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We can further simplify our proposal by assuming a Markov structure.

$$q(x_{0:t}) = q(x_0) \prod_{k=1}^{t} q(x_k|x_{k-1}).$$

This allows us to obtain purely recursive weight computation

$$\begin{aligned} W_{0:t}(x_{0:t}) &= \frac{\gamma(x_{0:t}, y_{1:t})}{q(x_{0:t})}, && (16)\\ &= \frac{\gamma(x_{0:t-1}, y_{1:t-1})}{q(x_{0:t-1})} \frac{\tau(x_t|x_{t-1})g(y_t|x_t)}{q(x_t|x_{t-1})}, && (17)\\ &= W_{0:t-1}(x_{0:t-1}) \frac{\tau(x_t|x_{t-1})g(y_t|x_t)}{q(x_t|x_{t-1})}, && (18)\\ &= W_{0:t-1}(x_{0:t-1}) W_t(x_t, x_{t-1}), && (19) \end{aligned}$$

Let us implement it. Assume that we have computed the unnormalised weights $\mathrm{W}_{0:t-1}^{(i)} = W(x_{0:t-1}^{(i)})$ recursively and obtained samples $x_{0:t-1}^{(i)}$. We only need the last sample $x_{t-1}^{(i)}$ to obtain the weight update given in (19). And also note that $\mathrm{W}_{0:t-1}^{(i)}$ for $i = 1, \ldots, N$ are just numbers, they do not need the storage of previous samples. We can now sample from the Markov proposal $x_t^{(i)} \sim q(x_t|x_{t-1}^{(i)})$ and compute the weights of the path sampler at time t as

$$\mathrm{W}_{1:t}^{(i)} = \mathrm{W}_{1:t-1}^{(i)} \times \mathrm{W}_t^{(i)},$$

where

$$\mathrm{W}_t^{(i)} = \frac{\tau(x_t^{(i)}|x_{t-1}^{(i)})g(y_t|x_t^{(i)})}{q(x_t^{(i)}|x_{t-1}^{(i)})}.$$

Given the samples $x_{t-1}^{(i)}$, we first perform sampling step

$$x_t^{(i)} \sim q(x_t|x_{t-1})$$

and then compute

$$\mathrm{W}_t^{(i)} = \frac{\tau(x_t^{(i)}|x_{t-1}^{(i)})g(y_t|x_t^{(i)})}{q(x_t^{(i)}|x_{t-1}^{(i)})}.$$

and update

$$\mathrm{W}_{0:t}^{(i)} = \mathrm{W}_{0:t-1}^{(i)} \times \mathrm{W}_t^{(i)}.$$

These are unnormalised weights and we normalise them to obtain,

$$\mathrm{w}_{1:t}^{(i)} = \frac{\mathrm{W}_{1:t}^{(i)}}{\sum_{i=1}^{N} \mathrm{W}_{1:t}^{(i)}},$$

which finally leads to the empirical measure,

$$\pi^N(\mathrm{d}x_{0:t}) = \sum_{i=1}^{N} \mathrm{w}_{1:t}^{(i)} \delta_{x_{0:t}^{(i)}}(\mathrm{d}x_{0:t}).$$

Particle filters

Sequential Importance Sampling (SIS)

- Sample $x_0^{(i)} \sim q(x_0)$ for $i = 1, \ldots, N$.
- For $t \geq 1$
  - Sample: $x_t^{(i)} \sim q(x_t | x_{t-1}^{(i)})$,
  - Compute weights:

$$\mathrm{W}_t^{(i)} = \frac{\tau(x_t^{(i)} | x_{t-1}^{(i)}) g(y_t | x_t^{(i)})}{q(x_t^{(i)} | x_{t-1}^{(i)})}.$$

and update

$$\mathrm{W}_{0:t}^{(i)} = \mathrm{W}_{0:t-1}^{(i)} \times \mathrm{W}_t^{(i)}.$$

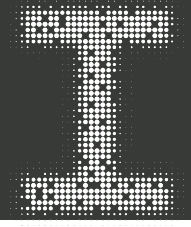
Normalise weights,

$$\mathrm{w}_{0:t}^{(i)} = \frac{\mathrm{W}_{0:t}^{(i)}}{\sum_{i=1}^{N} \mathrm{W}_{0:t}^{(i)}}.$$

  - Report

$$\pi_t^N(\mathrm{d}x_{0:t}) = \sum_{i=1}^{N} \mathrm{w}_{0:t}^{(i)} \delta_{x_{0:t}^{(i)}}(\mathrm{d}x_{0:t}).$$

There is a well-known problem with this scheme: *Weight degeneracy*.



There is a well-known problem with this scheme: *Weight degeneracy*.

To resolve this, the approach is to introduce resampling steps.

Particle filters

Sequential Importance Sampling - Resampling (SISR)

- Sample $x_0^{(i)} \sim q(x_0)$ for $i = 1, \ldots, N$.
- For $t \geq 1$
  - Sample: $\bar{x}_t^{(i)} \sim q(x_t | x_{t-1}^{(i)})$,
  - Compute weights:

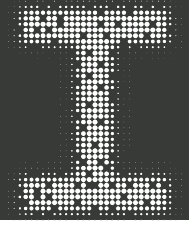
$$\mathrm{W}_t^{(i)} = \frac{\tau(\bar{x}_t^{(i)} | x_{t-1}^{(i)}) g(y_t | \bar{x}_t^{(i)})}{q(\bar{x}_t^{(i)} | x_{t-1}^{(i)})}.$$

    Normalise: $\mathrm{w}_t^{(i)} = \mathrm{W}_t^{(i)} / \sum_{i=1}^N \mathrm{W}_t^{(i)}$
  - Report

$$\tilde{\pi}_t^N(\mathrm{d}x_{0:t}) = \sum_{i=1}^N \mathrm{w}_{0:t}^{(i)} \delta_{\bar{x}_{0:t}^{(i)}}(\mathrm{d}x_{0:t}).$$

  - Resample:

$$x_t^{(i)} \sim \sum_{i=1}^N \mathrm{w}_t^{(i)} \delta_{\bar{x}_t^{(i)}}(\mathrm{d}x_t).$$



The bootstrap particle filter (BPF) is the SISR algorithm with the following choices:

$$q(x_t|x_{t-1}) = \tau(x_t|x_{t-1}),$$

Particle filters

Bootstrap particle filter

- Sample $x_0^{(i)} \sim q(x_0)$ for $i = 1, \ldots, N$.
- For $t \geq 1$
  - Sample: $\bar{x}_t^{(i)} \sim \tau(x_t | x_{t-1}^{(i)})$,
  - Compute weights:

$$\mathrm{W}_t^{(i)} = g(y_t | \bar{x}_t^{(i)}),$$

    Normalise: $\mathrm{w}_t^{(i)} = \mathrm{W}_t^{(i)} / \sum_{i=1}^N \mathrm{W}_t^{(i)}$
  - Report

$$\pi_t^N(\mathrm{d}x_{0:t}) = \sum_{i=1}^N \mathrm{w}_{1:t}^{(i)} \delta_{\bar{x}_{0:t}^{(i)}}(\mathrm{d}x_{0:t}).$$

  - Resample:

$$x_t^{(i)} \sim \sum_{i=1}^N \mathrm{w}_t^{(i)} \delta_{\bar{x}_t^{(i)}}(\mathrm{d}x_t).$$

References I